Worksheet 2/3 Solutions

MATH 33A

1. (a) Let $T : \mathbb{R} \to \mathbb{R}^2$ be a map defined by $T(x) = (x^2, x+1)$. To check if T is a linear transformation, we need to verify if $T(x+y) = T(x)+T(y)$ and $T(cx) = cT(x)$ for all $x, y \in \mathbb{R}$ and $c \in \mathbb{R}$.
For $T(x+y) = T(x) + T(y)$, we have:

$$T(x + y) = ((x + y)^2, (x + y) + 1) = (x^2 + 2xy + y^2, x + y + 1),$$

$$T(x) + T(y) = (x^2, x + 1) + (y^2, y + 1) = (x^2 + y^2, x + y + 2).$$

Since $(x^2 + 2xy + y^2, x + y + 1) \neq (x^2 + y^2, x + y + 2)$, T is not a linear transformation.

(b) Let $P : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $P(x, y) = (2x+y, x-y, 3y)$. To check if P is a linear transformation, we need to verify if $P(u + v) = P(u) + P(v)$ and $P(cu) = cP(u)$ for all $u, v \in \mathbb{R}^2$ and $c \in \mathbb{R}$.
For $P(u + v) = P(u) + P(v)$, we have:

$$P(x_1+x_2, y_1+y_2) = (2(x_1+x_2)+(y_1+y_2), (x_1+x_2)-(y_1+y_2), 3(y_1+y_2)),$$

$$P(x_1, y_1)+P(x_2, y_2) = (2x_1+y_1, x_1-y_1, 3y_1)+(2x_2+y_2, x_2-y_2, 3y_2) = (2(x_1+x_2)+(y_1+y_2), (x_1+x_2)-$$

For $P(cu) = cP(u)$, we have:

$$P(cx, cy) = (2(cx) + cy, cx - cy, 3(cy)),$$

$$cP(x, y) = c(2x + y, x - y, 3y) = (2(cx) + cy, cx - cy, 3(cy)).$$

Since $P(u + v) = P(u) + P(v)$ and $P(cu) = cP(u)$, P is a linear transformation.
The matrix associated with P is a 3×2 matrix:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

2. Show that the family of linear transformations $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

satisfy the following properties:

(a) They commute, i.e. $A_\theta A_{\theta'} = A_{\theta'} A_\theta$.

$$\begin{aligned} A_\theta A_{\theta'} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\theta' - \sin\theta\sin\theta' & -\cos\theta\sin\theta' - \sin\theta\cos\theta' \\ \sin\theta\cos\theta' + \cos\theta\sin\theta' & -\sin\theta\sin\theta' + \cos\theta\cos\theta' \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\theta') & -\sin(\theta+\theta') \\ \sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix} \\ &= A_{\theta+\theta'} \\ &= A_{\theta'} A_\theta. \end{aligned}$$

(b) They are 2π-periodic, i.e. $A_{\theta+2\pi} = A_\theta$.
Since $\cos(\theta+2\pi) = \cos\theta$ and $\sin(\theta+2\pi) = \sin\theta$, we have:

$$A_{\theta+2\pi} = \begin{bmatrix} \cos(\theta+2\pi) & -\sin(\theta+2\pi) \\ \sin(\theta+2\pi) & \cos(\theta+2\pi) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A_\theta.$$

(c) They rotate the unit vector e_1 by θ degrees to the vector $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

$$\begin{aligned} A_\theta e_1 &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \end{aligned}$$

3. (a) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A^{-1}.
To find the inverse of a 2×2 matrix, we can use the formula:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc \neq 0$.
In this case, $a = 1, b = 2, c = 3, d = 4$, and $\det(A) = 1\cdot4 - 2\cdot3 = -2$.
Thus,

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}.$$

To show that $AA^{-1} = A^{-1}A = I_2$, we compute the products:

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

$$A^{-1}A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Not every matrix has an inverse. A 2×2 matrix A is invertible if and only if its determinant $\det(A) = ad - bc \neq 0$.

(b) To solve the system $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where A is invertible, we can use the formula $x = A^{-1}b$.

$$x = A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.$$

(c) To show that $(AB)^{-1} = B^{-1}A^{-1}$, we compute the product $(AB)(B^{-1}A^{-1})$:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I. \end{aligned}$$

Similarly, we compute the product $(B^{-1}A^{-1})(AB)$:

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(AA^{-1})B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I. \end{aligned}$$

Therefore, $(AB)^{-1} = B^{-1}A^{-1}$.

(d) Suppose A is an $n \times n$ matrix such that all values in A^k are bounded by r^k for any k.

We need to show that $I_n - A$ is invertible and

$$(I_n - A)^{-1} = I + A + A^2 + ...$$

Note that the expression on the right is well-defined, since each entry is bounded in absolute value by $1 + r + r^2 + ...\frac{1}{1-r}$, which is finite, and therefore converges by the Absolute Convergence Test. Let

$$B = I + A + A^2 + A^3 + ...$$

Then $(I_n - A)B = B(I_n - A) = I_n$. To see this, note that

$$(I_n-A)B = (I_n-A)(I_n+A+A^2+A^3+...) = I_n+A+A^2+...-A^2-A^3-... = I_n,$$

and similarly,

$$B(I_n-A) = (I_n+A+A^2+A^3+...)(I_n-A) = I_n+A+A^2+...-A-A^2-... = I_n.$$

Since $(I_n - A)B = B(I_n - A) = I_n$, we have

$$(I_n - A)^{-1} = B = I + A + A^2 + ...$$

4. To find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ mapping the vector $\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}1\\2\end{bmatrix}$ and the vector $\begin{bmatrix}-2\\1\end{bmatrix} \to \begin{bmatrix}1\\1\end{bmatrix}$, we can write the transformation matrix T as follows:

$$T = \begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

We need to find the values of a, b, c, d such that:

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

Solving these two systems of equations, we get:

$$\begin{cases}a + b = 1\\c + d = 2\end{cases} \quad \text{and} \quad \begin{cases}-2a + b = 1\\-2c + d = 1\end{cases}.$$

Solving the first system, we get:

$$b = 1 - a \quad \text{and} \quad d = 2 - c.$$

Substituting these expressions into the second system, we obtain:

$$\begin{cases}-2a + 1 - a = 1\\-2c + 2 - c = 1\end{cases}.$$

Solving this system, we find $a = 0$ and $c = \frac{1}{3}$. Then, $b = 1$ and $d = \frac{5}{3}$. So, the transformation matrix is:

$$T = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{5}{3} \end{bmatrix}.$$

5. Show that $W \subset \mathbb{R}^3 = \{(x, y, z) : x + y + z = 0\}$ is a linear subspace. We need to show that W is closed under addition and scalar multiplication.

   Let $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2) \in W$. Then $x_1+y_1+z_1 = x_2+y_2+z_2 = 0$.

   For any scalar α, we have:

$$\alpha u = (\alpha x_1, \alpha y_1, \alpha z_1).$$

   We need to show that $\alpha u \in W$:

$$\alpha x_1 + \alpha y_1 + \alpha z_1 = \alpha(x_1 + y_1 + z_1) = \alpha(0) = 0.$$

   Thus, $\alpha u \in W$. Now we need to show that $u + v \in W$:

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

   We need to show that $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0$:

$$(x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1)+(x_2+y_2+z_2) = 0+0 = 0.$$

   Thus, $u + v \in W$. Since W is closed under addition and scalar multiplication, it is a linear subspace.

   To find a set of linearly independent vectors that span W, consider the vectors:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

   These vectors are linearly independent and satisfy the equation $x+y+z = 0$. Any linear combination of these vectors will also satisfy the equation and belong to W. Therefore, they span W.

   The dimension of W is the number of linearly independent vectors in the basis. In this case, the dimension of W is 2.